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Joint Distributions of Observables on Quantum Logics

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A joint distribution of a set of observables on a quantum logic in a state m is defined and its properties are derived. It is shown that if the joint distribution exists, then the observables can be represented in the state m by a set of commuting operators on a Hilbert space.

1. INTRODUCTION

For the study of simultaneous measurements of a set of observables the notion of their joint distribution is very important. Joint distributions of observables were studied by several authors, e.g., Moyal (1949), Urbanik (1961), Varadarajan (1962), Margenau (1963), Gudder (1968).

For the case of the logic $L(H)$, i.e., the set of all closed subspaces of a complex separable Hilbert space H, Urbanik (1961) and Varadarajan (1962) defined joint distributions in the following way. We define $\mu^{r,s}$ on $B(R)$ as

$$\mu^{r,s}(E) = \mu\{(t_1, t_2): rt_1 + st_2 \in E\}$$

where $E \in B(R)$, $(r, s) \in R^2$ and μ is a Borel measure on $B(R^2)$. We shall say that the observables x, y have a joint distribution in the state m if there is a Borel measure μ_m on $B(R^2)$ such that $\mu_m^{r,s}(E) = m[(rx + sy)(E)]$ for all $(r, s) \in R^2$ and $E \in B(R)$. Joint distributions of this type are called "type-2 joint distributions" (Gudder, 1968).

Another type of joint distributions—"type-1 joint distributions"—was defined by Gudder (1968) in the following way. We say that x and y have a type-1 joint distribution in a state m if there is a two-dimensional Borel measure $m_{x,y}$ such that $m_{x,y}(E \times F) = m[x(E) \wedge y(F)]$ for all $E, F \in B(R)$, and we call $m_{x,y}$ the joint distribution of x and y in the state m. Gudder considered a sum logic in that $x(E) \wedge y(F)$ exists for any $E, F \in B(R)$.

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In the present paper another type of joint distribution is defined and some of its properties are derived. This definition is close to that of the type-2 joint distributions, but a wider class of functions of x and y is considered.

2. LOGICS, STATES, OBSERVABLES

Let L be a partially ordered set with the first and last elements 0 and 1, respectively, which is closed under a complementation $a \mapsto a'$ satisfying

(i) $(a')' = a$,
(ii) $a \leqslant b$ implies $b' \leqslant a'$.

We denote the least upper bound and greatest lower bound of $a, b \in L$, if they exist, by $a \vee b$ and $a \wedge b$, respectively, and assume

(iii) $a \vee a' = 1$ for all $a \in L$.

We say that $a, b \in L$ are disjoint and write $a \perp b$ if $a \leqslant b'$. We say that $a, b \in L$ are compatible and write $a \leftrightarrow b$ if there exist mutually disjoint elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. We call L a logic if it also satisfies

(iv) $\vee a_i \in L$ for any disjoint sequence $\{a_i\} \subset L$,
(v) if $a, b, c \in L$ are mutually compatible, then $a \leftrightarrow b \vee c$.
(vi) $a \leqslant b$ implies $b = a \vee (b \wedge a')$.

A state is a nonnegative function m on L satisfying

(i) $m(1) = 1$,
(ii) $m(\vee a_i) = \sum m(a_i)$ for any disjoint sequence $\{a_i\} \subset L$.

A set M of states is full if $m(a) \leqslant m(b)$ for all $m \in M$ imply $a \leqslant b$, $a, b \in L$.

We shall call the couple (L, M), where L is a logic and M is a convex full set of states, the quantum logic. We shall in addition suppose that to any $a \in L$, $a \neq 0$, there is an $m \in M$ such that $m(a) = 1$.

An observable x is a σ-homomorphism from the Borel sets $B(R)$ of the real line R into L. A collection of observables $\{x_\lambda : \lambda \in \Lambda\}$ is compatible if $x_\lambda(E) \leftrightarrow x_\mu(F)$ for all $E, F \in B(R)$ and $\lambda, \mu \in \Delta$. If x is an observable and u is a Borel function on R, we define the observable $u(x)$ by $u(x)(E) = x[u^{-1}(E)]$ for all $E \in B(R)$. If ψ is an n-dimensional Borel function on R, we define the observable $\psi(u_1(x), \ldots, u_n(x))$ by $\psi(u_1(x), \ldots, u_n(x))(E) = x\{t: \psi(u_1(t), \ldots, u_n(t)) \in E\}$, $E \in B(R)$. The spectrum $\sigma(x)$ of an observable x is the smallest closed set E such that $x(E) = 1$. An observable x is bounded if $\sigma(x)$ is bounded. The expectation of an observable x in the state m is

$$m(x) = \int \lambda m[x(d\lambda)]$$

if the integral exists. The probability measure $m_x(E) = m[x(E)]$, $E \in B(R)$ is called the distribution of x in the state m. An observable x is a proposition observable if $\sigma(x) \subset \{0, 1\}$. The following statements are equivalent (Mackey, 1963):

(i) x is a proposition observable,
(ii) x is a characteristic function of an observable,
(iii) x is an idempotent, i.e., $x^2 = x$.

Let X_L be the set of all proposition observables on L. We define a partial ordering on X_L by setting that $x \leqslant y$ if $m(x) \leqslant m(y)$ for all $m \in M$ and an orthocomplementation by setting $x' = f(x)$, where $f(t) = 1 - t$, $t \in R$. To each $a \in L$ there is an observable $x_a \in X_L$ such that $x_a\{1\} = a$. The map $a \mapsto x_a$ from L onto X_L is an isomorphism.

Let X be the set of all bounded observables on L. We say that $z \in X$ is the sum of x and y from X if $m(z) = m(x) + m(y)$ for all $m \in M$. In this case we write $z = x + y$. We shall suppose that $x + y$ exists for all $x, y \in X$ and that it is unique, so that $m(x) = m(y)$ for all $m \in M$ implies $x = y$ (Gudder, 1966). Let us denote by I the unique observable on L such that $m(I) = 1$ for all $m \in M$.

3. JOINT DISTRIBUTIONS OF OBSERVABLES

The set X of all bounded observables on the quantum logic (L, M) is closed under the formations of bounded functions of observables and it is supposed to be closed under the formations of sums of observables.

Let $x_1, \ldots, x_k \in X$. Let us denote by $S(x_1, \ldots, x_k)$ the smallest subset of X closed under the formations of sums and bounded Borel functions of observables which contains $x_1, \ldots, x_k$ and the identity observable I. Let $y \in S(x_1, \ldots, x_k)$ be of the form $y = f(x_1, \ldots, x_k)$, where f is a "function" of $x_1, \ldots, x_k, I$. f can be of the form

$$f(x_1, \ldots, x_k) = c_0 I + \sum_{i=1}^{k} x_i c_i, \; c_0, c_1, \ldots, c_k \in R$$

or

$$f(x_1, \ldots, x_k) = g\left(c_0 I + \sum_{i=1}^{k} c_i x_i\right)$$

etc. Let us denote by $\tilde{f}(t_1, \ldots, t_k)$ the function $\tilde{f}: R^k \to R$ obtained from f by replacing $x_1, \ldots, x_k$ by real numbers $t_1, \ldots, t_k$.

Definition 1. We shall say that the observables $x_1, \ldots, x_k \in X$ have a joint distribution in a state $m \in M$ if there is a measure μ on $B(R^k)$ such that for any $y = f(x_1, \ldots, x_k) \in S(x_1, \ldots, x_k)$ we have

$$m(y) = \int \lambda \mu f^{-1}(d\lambda)$$

and we shall call μ the joint distribution of $x_1, \ldots, x_k$ in the state m.

If μ is the joint distribution of $x_1, \ldots, x_k$ in a state m, then

$$\begin{aligned} m[f(x_1, \ldots, x_k)(E)] &= m\{\chi_E[f(x_1, \ldots, x_k)]\} = \int \lambda \mu(\chi_E f)^{-1}(d\lambda) \\ &= \int \lambda(\mu \tilde{f}^{-1} \chi_E^{-1})(d\lambda) = \int \chi_E(\lambda) \mu \tilde{f}^{-1}(d\lambda) \\ &= \mu[\tilde{f}^{-1}(E)] \end{aligned}$$

Theorem 1. Let the joint distribution exist for the observables $x, y \in X$ in the state m. Then $\mu(E \times F) = m[\chi_E(x) \circ \chi_F(y)]$ for all $E, F \in B(R)$, where $\circ$ denotes the pseudoproduct defined by $u \circ v = \frac{1}{4}[(u + v)^2 - (u - v)^2]$, $u, v \in X$.

Proof.

$$\chi_E(x) \circ \chi_F(y) = \tfrac{1}{4}\{[\chi_E(x) + \chi_F(y)]^2 - [\chi_E(x) - \chi_F(y)]^2\} \in S(x, y)$$

so that

$$\begin{aligned} m[\chi_E(x) \circ \chi_F(y)] &= \int \tfrac{1}{4}\{[\chi_E(t_1) + \chi_F(t_2)]^2 - [\chi_E(t_1) - \chi_F(t_2)]^2\}\, d\mu(t_1, t_2) \\ &= \int \chi_E(t_1)\chi_F(t_2)\, d\mu(t_1, t_2) = \int \chi_{E \times F}(t_1, t_2)\, d\mu \\ &= \mu(E \times F) \end{aligned}$$

Corollary. If the joint distribution of x, y in the state m exists, then it is uniquely defined.

Proof. The set function defined by $\mu(E \times F) = m[\chi_E(x) \circ \chi_F(y)]$ on all rectangle sets $E \times F \in B(R^2)$ can be uniquely extended to a measure μ on $B(R^2)$ (Halmos, 1974).

Theorem 2. The observables $x, y \in X$ have a joint distribution in each state $m \in M$ if and only if they are compatible.

Proof. Let $x \leftrightarrow y$, then there exists a σ homomorphism $z: B(R^2) \to L$ such that

$$x(E) = z[\pi_1^{-1}(E)], \qquad y(E) = z[\pi_2^{-1}(E)]$$

where π_1, π_2 are the projections $\pi_i(\omega_1, \omega_2) = \omega_i$, $i = 1, 2$ (Varadarajan, 1962). For any measurable function $f(\omega_1, \omega_2)$ we have

$$\begin{aligned} f(x, y)(E) &= f(\pi_1(z), \pi_2(z))(E) \\ &= z\{\omega = (\omega_1, \omega_2): f(\pi_1(\omega), \pi_2(\omega)) \in E\} \\ &= z\{\omega: f(\omega_1, \omega_2) \in E\} = z[f^{-1}(E)], \qquad E \in B(R) \end{aligned}$$

For any $m \in M$ then $m[f(x, y)] = m[f(z)] = \lambda m_z f^{-1}(d\lambda)$, where $m_z(E) = m[z(E)]$, so that $\mu E = m_z(E)$, $E \in B(R^2)$, is the joint distribution in the state m.

Now let joint distributions exist in all $m \in M$. For $m \in M$, let μ_m be the joint distribution. Then

$$\mu_m(E \times F) = m[\chi_E(x) \circ \chi_F(y)]$$

on all rectangles $E \times F \in B(R^2)$. First we shall show that $\chi_E(x) \circ \chi_F(y)$ is a simple observable, i.e., that

$$[\chi_E(x) \circ \chi_F(y)]^2 = \chi_E(x) \circ \chi_F(y)$$

Indeed,

$$m\{[\chi_E(x) \circ \chi_F(y)]^2\} = \int \lambda \mu_m \tilde{f}^{-1}(d\lambda)$$

where

$$\tilde{f}(t_1, t_2) = [\chi_E(t_1) \circ \chi_F(t_2)]^2 = \chi_{E \times F}^2(t_1, t_2) = \chi_{E \times F}(t_1, t_2)$$

so that

$$\begin{aligned} m\{[\chi_E(x) \circ \chi_F(y)]^2\} &= \int \chi_{E \times F}(t_1, t_2)\, d\mu_m(t_1, t_2) \\ &= \mu_m(E \times F) \\ &= m[\chi_E(x) \circ \chi_F(y)] \end{aligned}$$

From this it follows that

$$[\chi_E(x) \circ \chi_F(y)]^2 = \chi_E(x) \circ \chi_F(y)$$

Now let $E \times F \cap G \times H = \varnothing$, $E, F, G, H \in B(R)$. Then for any $m \in M$,

$$m[\chi_E(x) \circ \chi_F(y) + \chi_G(x) \circ \chi_H(y)] = \int \lambda \mu_m \tilde{f}^{-1}(d\lambda)$$

where

$$\tilde{f}(t_1, t_2) = \chi_E(t_1)\chi_F(t_2) + \chi_G(t_1)\chi_H(t_2) = \chi_{E\times F}(t_1, t_2) + \chi_{G\times H}(t_1, t_2)$$
$$= \chi_{E\times F\cup G\times H}(t_1, t_2)$$

We can show as above that $\chi_E(x) \circ \chi_F(y) + \chi_G(x) \circ \chi_H(y)$ is an idempotent. But then

$$\chi_E(x) \circ \chi_F(y) + \chi_G(x) \circ \chi_H(y) = \chi_E(x) \circ \chi_F(y) \vee \chi_G(x) \circ \chi_H(y)$$

(Gudder, 1966). Clearly,

$$m[\chi_E(x) \circ \chi_F(y) + \chi_G(x) \circ \chi_H(y)] = \mu_m(E \times F \cup G \times H)$$

Let $\mathscr{A}$ be the algebra consisting of all finite disjoint unions of measurable rectangles in R^2. Let us define

$$h\left(\bigcup_{i=1}^{n} E_i \times F_i\right) = \bigvee_{i=1}^{n} \chi_{E_i}(x) \circ \chi_{F_i}(y)$$

where $E_i \times F_i \cap E_j \times F_j = \varnothing$ for $i \neq j$, $i, j = 1, 2, \ldots, n$. If

$$\bigcup_{i=1}^{\infty} E_i \times F_i = E \times F$$

where $E_i \times F_i \cap E_j \times F_j = \varnothing$ for $i \neq j$, then

$$\sum_{i=1}^{\infty} m[h(E_i \times F_i)] = m\left[\bigvee_{i=1}^{\infty} \chi_E(x) \circ \chi_F(y)\right] = \int \lambda \mu_m \tilde{f}^{-1}(d\lambda)$$

where

$$\tilde{f}(t_1, t_2) = \bigvee_{i=1}^{\infty} \chi_{E_i}(t_1)\chi_{F_i}(t_2) = \sum_{i=1}^{\infty} \chi_{E_i}(t_1)\chi_{F_i}(t_2)$$
$$= \chi_{\bigcup_{i=1}^{\infty} E_i\times F_i}(t_1, t_2) = \chi_{E\times F}(t_1, t_2)$$

Thus

$$m\left(\bigvee_{i=1}^{\infty} h(E_i \times F_i)\right) = \sum_{i=1}^{\infty} m[h(E_i \times F_i)] = \mu_m(E \times F) = m[h(E \times F)]$$

for all $m \in M$. From this it follows that

$$\bigvee_{i=1}^{\infty} h(E_i \times F_i) = h(E \times F)$$

i.e., h is a σ homomorphism from $\mathscr{A}$ to L. As

$$h(E \times R) = \chi_E(x) \circ \chi_R(y) = \chi_E(x) \circ I = \chi_E(x)$$

and analogically, $h(R \times F) = \chi_F(y)$, we get that $\chi_E(x) \leftrightarrow \chi_F(y)$ for all $E, F \in B(R)$, i.e., $x \leftrightarrow y$.

Theorems 1 and 2 can be generalized to any finite set of bounded observables. For any indexed set of observables we can define the following. An indexed set $\{x_\lambda, \lambda \in \Lambda\}$ of bounded observables will be said to have a joint distribution in a state m if for any k and $\lambda_1, \ldots, \lambda_k \in \Delta$, $x_{\lambda 1}, \ldots, x_{\lambda k}$ have a joint distribution in the state m.

Now we shall show that from the existence of a joint distribution in a state m it follows that the observables have a compatibility property relative to the state m. To show this we need a definition and a lemma.

Definition 2. Let $x, y \in X$ and $m \in M$. We shall say that $x = y$ modulo m and write $x = y[m]$ if $m[(x - y)^2] = 0$.

From the Schwarz inequality it follows that

$$[m(x - y)]^2 \leqslant m[(x - y)^2]$$

so that $x = y[m]$implies $m(x) = m(y)$.

Let $x_1, \ldots, x_k \in X$ have a joint distribution in a state m. Let $x, y, z \in S(x_1, \ldots, x_k)$ such that

$$m(x) = \int \lambda\mu \tilde{f}_1^{-1}(d\lambda), \qquad m(y) = \int \lambda\mu \tilde{f}_2^{-1}(d\lambda), \qquad m(z) = \int \lambda\mu \tilde{f}_3^{-1}(d\lambda)$$

Then

$$m([(x \circ y) \circ z - x \circ (y \circ z)]^2) = 0, \qquad m([(x + y) \circ z - (x \circ z + y \circ z)]^2) = 0$$

$$m([(\alpha x) \circ y - \alpha(x \circ y)]^2) = 0$$

for any $\alpha \in R$. Thus we get the following statement.

Lemma 1. If the observables $x_1, \ldots, x_k$ have a joint distribution in the state m, then the pseudoproduct $\circ$ is associative, distributive (relative to addition), and homogeneous (relative to scalar multiplication) modulo m on all $f(x_1, \ldots, x_k) \in S(x_1, \ldots, x_k)$.

Theorem 3. Let $x_1, \ldots, x_k \in X$ and $m \in M$. Let S_0 be the smallest subset of X closed under the formations of finite linear combinations and pseudoproducts of observables which contains $x_1, \ldots, x_k$ and I. Let the pseudoproduct be distributive, associative, and homogeneous on S_0 modulo m, i.e., if $x, y, z \in S_0$, $\alpha \in R$, then

$$m\{[(x + y) \circ z - (x \circ z + y \circ z)]^2\} = 0$$

$$m\{[(\alpha x) \circ y - \alpha(x \circ y)]^2\} = 0$$

$$m\{[(x \circ y) \circ z - x \circ (y \circ z)]^2\} = 0$$

Then there is a real Hilbert space H and a map $x \mapsto \pi_x$ from S_0 into the set $B(H)$ of all bounded operators on H such that $\pi_x \pi_y = \pi_y \pi_x$ for all $x, y \in S_0$, and $m(x) = (\phi_0, \pi_x \phi_0)$ for all $x \in S_0$ with a $\phi_0 \in H$.

Proof. Let $\alpha, \beta \in R$. From the distributivity modulo m we have

$$m[(\alpha x + \beta y)^2] = \alpha^2 m(x^2) + \beta^2 m(y^2) + 2\alpha\beta m(x \circ y) \geqslant 0$$

i.e.,

$$\frac{\alpha^2}{\beta^2} m(x^2) + 2 \frac{\alpha}{\beta} m(x \circ y) + m(y^2) \geqslant 0$$

From this we get

$$[m(x \circ y)]^2 \leqslant m(x^2) m(y^2)$$

Then

$$m[(x + y)^2] \leqslant \{[m(x^2)]^{1/2} + [m(y^2)]^{1/2}\}^2$$

i.e., $[m(x^2)]^{1/2}$ is a seminorm on the linear space S_0. Let us write $x \sim y$ whenever $m[(x - y)^2] = 0$ and replace S_0 by the set $\hat{S}_0$ of all equivalence classes with respect to the relation $\sim$. Let $[x] \in \hat{S}_0$ be the class containing $x \in S_0$. We define addition and multiplication on S_0 by $\alpha[x] + \beta[y] = [\alpha x + \beta y]$ and $[x][y] = [x \circ y]$. These operations do not depend on the choice of the representants. Indeed, let $x_1, x_2 \in [x]$ and $y_1, y_2 \in [y]$. Then

$$\{m[(x_1 + y_1) - (x_2 + y_2)]^2\}^{1/2} \leqslant \{m[(x_1 - x_2)^2]\}^{1/2} + \{m[(y_1 - y_2)^2]\}^{1/2} = 0$$

and

$$\begin{aligned}\{m[(x_1 \circ y_1 - x_2 \circ y_2)^2]\}^{1/2} &= \{m[(x_1 - x_2) \circ y_1 + x_2 \circ (y_1 - y_2)]^2\}^{1/2} \\ &\leqslant \{m[(x_1 - x_2) \circ y_1]^2\}^{1/2} \\ &\quad + \{m[x_2 \circ (y_1 - y_2)]^2\}^{1/2} \\ &\leqslant \{m[(x_1 - x_2)^4] m(y_1{}^4)\}^{1/4} \\ &\quad + \{m(x_2{}^4) m[(y_1 - y_2)^4]\}^{1/4} = 0\end{aligned}$$

because $m(x^2) = 0$ implies $m_x(\{0\}) = 1$, so that

$$m(x^4) = \int \lambda^4 m_x(d\lambda) = 0$$

Then $\hat{S}_0$ with the operations $+$ and $\circ$ is a commutative real algebra. Further we shall proceed by the Gelfand–Naimark–Segal construction (Naimark, 1968, § 17). The map $[x], [y] \mapsto ([x], [y]) = m(x \circ y)$ is a symmetric linear functional on $\hat{S}_0$ and $m(x^2) = 0$ iff $[x] = 0$. The function $([x], [y])$ with $[x]$ and $[y]$ ranging over $\hat{S}_0$ has the usual properties of an inner product. Next, S_0 can be completed to a real Hilbert space H relative to the given inner product. For $x \in S_0$, we set $\pi_x{}^0$ for the operator on $\hat{S}_0$ defined by $\pi_x{}^0[y] = [x \circ y]$. Let $x \in S_0$, then

$$\|x\| = \sup\{|m(x)| : m \in M\} = \sup\{t : t \in \sigma(x)\}$$

and $\sigma(x^2) = [\sigma(x)]^2$ implies that $\|x^2\| = \|x\|^2$ (Gudder, 1965). Then for all $m' \in M$, $m'(\|x\|^2 I - x^2) \geqslant 0$, so that there is a $z \in X$ such that $z^2 = \|x\|^2 I - x^2$. From this it follows that

$$m[y^2 \circ (\|x\|^2 I - x^2)] = m(y^2 \circ z^2) \geqslant 0$$

i.e.,

$$m(y^2 \circ x^2) \leqslant \|x\|^2 m(y^2)$$

so that

$$\|\pi_x{}^0\|^2 = \sup_{[y]} \frac{\|\pi_x{}^0[y]\|^2}{\|[y]\|^2} = \sup_y \frac{m[(x \circ y)^2]}{m(y^2)} \leqslant \|x\|^2$$

Thus $\pi_x{}^0$ can be uniquely extended to an operator on H. Let us denote by π_x this extension. Now let $x, y, z \in S_0$ and $\alpha, \beta \in R$, then

$$\pi_x \pi_y[z] = \pi_x[y \circ z] = [x \circ y \circ z] = \pi_{x \circ y}[z] = \pi_{y \circ x}[z] = \pi_y \pi_x[z]$$

$$\pi_{\alpha x + \beta y}[z] = [(\alpha x + \beta y) \circ z] = \alpha[x \circ z] + \beta[y \circ z] = \alpha\pi_x[z] + \beta\pi_y[z]$$

so that $\pi_{\alpha x + \beta y} = \alpha\pi_x + \beta\pi_y$ and $\pi_{xy} = \pi_x\pi_y = \pi_y\pi_x$. Let $\phi_0 \in H$ be the class $[I]$. Then $(\phi_0, \pi_x\phi_0) = m(I \circ x) = m(x)$ for all $x \in S_0$.

In the following theorems we shall treat the relation between our joint distributions and the type-1 joint distributions (Gudder, 1968).

Theorem 4. If $x \leftrightarrow y$, then $\chi_E(x) \circ \chi_F(y) = \chi_E(x) \wedge \chi_F(y)$.

Proof. $x \leftrightarrow y$ implies that there is an observable u and real Borel functions f_1, f_2 such that $x = f_1(u)$, $y = f_2(u)$ (Gudder, 1965). Then

$$\chi_E(x) \circ \chi_F(y) = \chi_E(f_1(u)) \circ \chi_F(f_2(u)) = \chi_{f_1^{-1}(E)}(u)\chi_{f_2^{-1}(F)}(u)$$

$$= \chi_{f_1^{-1}(E)}(u) \wedge \chi_{f_2^{-1}(F)}(u) = \chi_E(x) \wedge \chi_F(y)$$

From Theorems 1, 2, and 4 it then follows that for compatible observables both joint distributions exist and are identical.

A logic is quite full if the statement $m(b) = 1$, whenever $m(a) = 1$ implies the statement $a \leqslant b$, $a, b \in L$ (Gudder, 1966).

Theorem 5. Let x, y be bounded observables on a quite full logic. If the joint distribution in a state m exists, then $\mu(E \times F) = \mu[x(E) \wedge y(F)]$, $E, F \in B(R)$.

Proof. If the joint distribution exists, then

$$m[(\chi_E(x) + \chi_F(y)]\{2\}) = \mu([\chi_E(t_1) + \chi_F(t_2)]^{-1}\{2\})$$
$$= \mu\{(t_1, t_2): \chi_E(t_1) + \chi_F(t_2) = 2\} = \mu(E \times F)$$

On the other hand, on a quite full logic,

$$[\chi_E(x) + \chi_F(y)](\{2\}) = x(E) \wedge y(F)$$

(Gudder, 1966), so that $m[x(E) \wedge y(F)] = \mu(E \times F)$.

Theorem 6. Let $L(H)$ be the logic of all closed subspaces of a complex separable Hilbert space H. If the type-1 joint distribution exists for the observables x, y in a state m, then

$$\mu(E \times F) = m[x(E) \wedge y(F)] = m[x(E) \circ y(F)]$$

Proof. Any state m on $L(H)$ can be written in the form $m = \sum r_i(\Phi_i, \cdot \Phi_i)$, where $\{\Phi_i\}$ is an orthonormal set of vectors in H and $r_i \geqslant 0$, $\sum r_i = 1$. Gudder (1968, Theorem 3.7) has proved that the type-1 joint distribution exists if and only if $x(E)y(F)\Phi_i = y(F)x(E)\Phi_i$, $E, F \in B(R)$, $i = 1, 2, \ldots$. Then $x(E)y(F)\Phi_i = y(F)x(E)\Phi_i = x(E) \wedge y(F)\Phi_i$, so that $m[x(E) \wedge y(F)] = \sum r_i(\Phi_i, x(E) \wedge y(F)\Phi_i) = \sum r_i(\Phi_i, x(E)y(F)\Phi_i) = \sum r_i(\Phi_i, x(E) \circ y(F)\Phi_i) = m[x(E) \circ y(F)]$.

We note that from the existence of a measure μ defined by $\mu(E \times F) = m[\chi_E(x) \circ \chi_F(y)]$, $E, F \in B(R)$ it need not follow, in general, that there is a joint distribution in the sense of Definition 1.

Theorem 7. Suppose that x and y are self-adjoint operators with a pure point spectrum. Let Φ_i, $i = 1, 2, \ldots$ be the common eigenvectors of x and y. Then there exists a joint distribution of x and y in the state $m = \sum r_i(\Phi_i, \cdot \Phi_i)$.

Proof. Let λ_i, μ_i be the eigenvalues of x and y, respectively, corresponding to the eigenstates Φ_i, $i = 1, 2, \ldots$. Then

$$f(x)\Phi_i = f(\lambda_i)\Phi_i, \qquad (x + y)\Phi_i = (\lambda_i + \mu_i)\Phi_i$$

for any Borel function f, so that

$$f(x, y)\Phi_i = \tilde{f}(\lambda_i, \mu_i)\Phi_i$$

for each $f(x, y) \in S(x, y)$. Let us define the measure μ by

$$\mu(G) = \sum \{r_i: (\lambda_i, \mu_i) \in G\}, G \in B(R^2)$$

Then

$$m[x(E) \circ y(F)] = m[x(E) \wedge y(F)] = \sum \{r_i : (\lambda_i, \mu_i) \in E \times F\} = \mu(E \times F)$$

and

$$m[f(x, y)] = \sum r_i \tilde{f}(\lambda_i, \mu_i) = \int_{R^2} \tilde{f}(t_1, t_2)\, d\mu(t_1, t_2)$$

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